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ESTIMATES RELATED TO SUMFREE SUBSETS OF SETS OF INTEGERS

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ABSTRACT

A subset A of the positive integers $\mathbb{Z}_+$ is called sumfree provided $(A + A) \cap A = \emptyset$. It is shown that any finite subset B of $\mathbb{Z}_+$ contains a sumfree subset A such that $|A| \geq \frac{1}{3}(|B| + 2)$, which is a slight improvement of earlier results of P. Erdös [Erd] and N. Alon–D. Kleitman [A-K]. The method used is harmonic analysis, refining the original approach of Erdös. In general, define $s_k(B)$ as the maximum size of a k-sumfree subset A of B, i.e. $(A)_k = \underbrace{A + \cdots + A}_{k \text{ times}}$ is disjoint from A. Elaborating the techniques permits one to prove that, for instance, $s_3(B) > \frac{|B|}{4} + c\frac{\log|B|}{\log\log|B|}$ as an improvement of the estimate $s_k(B) > \frac{|B|}{4}$ resulting from Erdös' argument. It is also shown that in an inequality $s_k(B) > \delta_k|B|$, valid for any finite subset B of $\mathbb{Z}_+$, necessarily $\delta_k \to 0$ for $k \to \infty$ (which seemed to be an unclear issue). The most interesting part of the paper are the methods we believe and the resulting harmonic analysis questions. They may be worthwhile to pursue.

1. Introduction

Call a subset A of $\mathbb{Z}_+$ sumfree provided $(A + A) \cap A = \emptyset$. It is observed in [Erd] that any finite subset B of $\mathbb{Z}$ contains a sumfree subset A such that

$$|A| \geq \frac{1}{3}\,|B|. \tag{1.1}$$

In [A-K], the authors pointed out that in fact the argument yields

$$|A| > \frac{1}{3}\,|B|, \text{ hence } |A| \geq \frac{1}{3}\,(|B| + 1). \tag{1.2}$$

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The purpose of this note is to develop a harmonic analysis approach to this and similar problems, estimating discrepancies using techniques related to trigonometric sums. We show, for instance, the following slight improvement of (1.2).

PROPOSITION 1.3:

$$S(B) \geq \frac{1}{3}\left(|B|+2\right), \quad \textit{for any } B \subset \mathbb{Z}_+.$$

$S(B)$ denotes the maximum size of a sumfree subset of B.

There is also the following fact, which in many cases yields a more significant improvement.

PROPOSITION 1.4:

$$S(B) \geq \frac{|B|}{3} + c_1 \left(\log |B|\right)^{-1} \left\| \sum_{k\in B} \cos 2\pi k\theta \right\|_1 .$$

Here c_1 is some fixed constant. From the solution to Littlewood's conjecture, one has always

$$\left\| \sum_{k\in B} e^{2\pi i k\theta} \right\|_1 \equiv \int_0^1 \left| \sum_{k\in B} \cos 2\pi k\theta \right| d\theta > c_2 \log |B| \tag{1.5}$$

(see [M-P-S] for the proof).

One may similarly define $S_3(B)$ as the size of the largest subset A of B satisfying

$$(A+A+A) \cap A = \emptyset. \tag{1.6}$$

The harmonic analysis techniques may be here exploited in a more successful way. The following improvement on the "obvious" inequality $S_3(B) > |B|/4$ is obtained.

PROPOSITION 1.7:

$$S_3(B) > \frac{|B|}{4} + c\, \frac{\log |B|}{\log\log |B|}.$$

From a technical point of view, this is the most interesting part of the paper.

2. Harmonic analysis formulation

Denote by f the indicator function of the arc $J = \left]\frac{1}{2}-\frac{1}{6},\ \frac{1}{2}+\frac{1}{6}\right[$ on the circle $\mathbb{T}=\mathbb{R}/\mathbb{Z}$.

Observe that if $A \subset \mathbb{Z}$ and $x \in \mathbb{T}$ such that $nx \in J$ for all $n \in A$, then clearly A is sumfree. Hence

$$S(B) \geq \max_{x\in\mathbb{T}} \sum_{m\in B} f(mx) = \frac{|B|}{3} + \max_{x\in\mathbb{T}} \sum_{m\in B} \left(f-\frac{1}{3}\right)(mx). \tag{2.1}$$

The Fourier expansion of the function $f-\frac{1}{3}$ yields

$$f(x)-\frac{1}{3} = \sum_{n\neq 0} \frac{(-1)^n}{n\pi}\cdot \sin\frac{n\pi}{3}\cdot e^{2\pi i n x} = -\frac{\sqrt{3}}{\pi}\sum_{n\geq 1}\frac{\chi(n)}{n}\cos nx \tag{2.2}$$

where χ is the multiplicative character defined by

$$\chi(n) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 3, \\ 1 & \text{if } n = 1 \pmod 3, \\ -1 & \text{if } n = 2 \pmod 3. \end{cases} \tag{2.3}$$

Denote by μ the Moebius function. Let $2, 3, 5, \ldots, P, \ldots$ be an enumeration of the consecutive primes. For a subset A of $\mathbb{Z}$, denote

$$\mathcal{G}_A = \{n \in \mathbb{Z} \mid (n,k) = 1 \text{ for all } k \in A\}.$$

Observe that

$$\sum_{k|P!,\,k|n} \mu(k) = \begin{cases} 1 & \text{if } n \in \mathcal{G}_{2,\ldots,P}, \\ 0 & \text{if } n \notin \mathcal{G}_{2,\ldots,P}. \end{cases} \tag{2.4}$$

Hence, one gets

$$\begin{aligned} \sum_{k|P!} \frac{\mu(k)}{k}\,\chi(k)\left(f-\frac{1}{3}\right)(kx) &= -\frac{\sqrt{3}}{\pi}\sum_{n\in\mathcal{G}_{2,\ldots,P}} \frac{\chi(n)}{n}\cos nx \\ &= -\frac{\sqrt{3}}{\pi}\cos x - \frac{\sqrt{3}}{\pi}\sum_{\substack{n\in\mathcal{G}_{2,\ldots,P}\\ n>1}} \frac{\chi(n)}{n}\cos nx \end{aligned} \tag{2.5}$$

and

$$\sum_{k|P!}\frac{\mu(k)}{k}\,\chi(k)\left[\sum_{m\in B}\left(f-\frac{1}{3}\right)(mkx)\right] =$$

$$-\frac{\sqrt{3}}{\pi}\sum_{m\in B}\cos mx - \frac{\sqrt{3}}{\pi}\sum_{\substack{n\in\mathcal{G}_{2,\ldots,P},\,n>1\\ m\in B}} \frac{\chi(n)}{n}\cos mnx. \tag{2.6}$$

3. Proof of Proposition 1.3

We minorate

$$\max_{x\in\mathbb{T}} \sum_{m\in B} \left(f - \frac{1}{3}\right)(mx) = \max_{x\in\mathbb{T}} \left[-\frac{\sqrt{3}}{\pi} \sum_{n\geq 1, m\in B} \frac{\chi(n)}{n} \cos nmx \right]. \tag{3.1}$$

Denote $0 < m_1 < m_2 < m_3 < \cdots < m_N$ the elements of B. We may assume $\gcd(B) = 1$. We distinguish first the cases $m_1 > 1$ and $m_1 = 1$.

(I): *Case* $m_1 > 1$. Define $j = \min\{j' = 2, \dots, N \mid m_{j'} \notin m_1\mathbb{Z}\}$. Denote for convenience

$$F(x) = -\sum_{n\geq 1, m\in B} \frac{\chi(n)}{n} \cos nmx. \tag{3.2}$$

Consider the test function

$$G(x) = (1 - \cos m_1 x)(1 - \cos m_j x) \tag{3.3}$$

satisfying $G \geq 0$, $\int_{\mathbb{T}} G = 1$. Hence

$$\max_{x\in\mathbb{T}} F(x) \geq \langle F, G\rangle = -\widehat{F}(m_1) - \widehat{F}(m_j) + \frac{1}{2}\,\widehat{F}(m_j - m_1) + \frac{1}{2}\widehat{F}(m_j + m_1). \tag{3.4}$$

By definition of j, we have

$$m_j \notin nB \quad \text{for } n > 1, \quad \text{hence } \widehat{F}(m_j) = -\frac{1}{2}, \tag{3.5}$$

$$m_j - m_1 \notin nB \quad \text{for } n \geq 1, \quad \text{hence } \widehat{F}(m_j - m_1) = 0, \tag{3.6}$$

$$m_j + m_1 \notin nB \quad \text{for } n > 1, \quad \text{hence } \widehat{F}(m_j + m_1) = 0 \quad \text{or} \quad -\frac{1}{2}. \tag{3.7}$$

For instance, if $m_j + m_1 = nm_{j'}$ with $n \geq 2$, there would follow $m_{j'} \leq \frac{1}{2}(m_1 + m_j) < m_j$, hence $j' < j$ and $m_{j'} \in m_1\mathbb{Z}$, $m_j \in m_1\mathbb{Z}$ (contradiction).

From (3.4)–(3.7), we get thus

$$\max_{x\in\mathbb{T}} F(x) \geq \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4} \tag{3.8}$$

and hence

$$(3.1) \geq \frac{\sqrt{3}}{\pi} \cdot \frac{3}{4} = 0,41\ldots > \frac{1}{3}. \tag{3.9}$$

(II): *Case* $m_1 = 1$. Distinguish the cases $m_2 > 2$ and $m_2 = 2$.

CASE $m_2 > 2$: Take

$$G(x) = 1 - \frac{4}{3} \cos x + \frac{1}{3} \cos 2x \geq 0. \tag{3.10}$$

Hence

$$\max_{x \in \mathbb{T}} F(x) \geq \langle F, G \rangle = -\frac{4}{3} \widehat{F}(1) + \frac{1}{3} \widehat{F}(2) = \frac{2}{3} + \frac{1}{12} = \frac{3}{4} \tag{3.11}$$

and

$$(3.1) \geq \frac{\sqrt{3}}{\pi} \cdot \frac{3}{4} > \frac{1}{3}. \tag{3.12}$$

CASE $m_2 = 2$: Thus $m_1 = 1$, $m_2 = 2$. Take

$$G(x) = (1 - \cos x)(1 - \cos m_3 x). \tag{3.13}$$

This yields

$$\max_{x \in \mathbb{T}} F(x) \geq \langle F, G \rangle = \frac{1}{2} - \widehat{F}(m_3) + \frac{1}{2} \widehat{F}(m_3 - 1) + \frac{1}{2} \widehat{F}(m_3 + 1). \tag{3.14}$$

(1): $m_3 = 3$. Then $\widehat{F}(m_3) = -\frac{1}{2}$, $\widehat{F}(m_3 - 1) = -\frac{1}{4}$, $\widehat{F}(m_3 + 1) \geq -\frac{3}{8}$ and

$$(3.14) \geq \frac{1}{2} + \frac{1}{2} - \frac{1}{8} - \frac{3}{16} = \frac{11}{16},$$

$$(3.1) \geq \frac{\sqrt{3}}{\pi} \cdot \frac{11}{16} = 0,37 \ldots > \frac{1}{3}. \tag{3.15}$$

(2): $m_3 = 4$. Then $\widehat{F}(m_3) = -\frac{3}{8}$, $\widehat{F}(m_3 - 1) = 0$, $\widehat{F}(m_3 + 1) \geq -\frac{2}{5}$ and

$$(3.14) \geq \frac{1}{2} + \frac{3}{8} - \frac{1}{5} = \frac{27}{40},$$

$$(3.1) \geq \frac{\sqrt{3}}{\pi} \cdot \frac{27}{40} = 0,37 \ldots > \frac{1}{3}. \tag{3.16}$$

(3): $m_3 = 5$. Then $\widehat{F}(m_3) = -\frac{2}{5}$, $\widehat{F}(m_3 - 1) = \frac{1}{8}$, $\widehat{F}(m_3 + 1) \geq -\frac{1}{2}$ and

$$(3.14) \geq \frac{1}{2} + \frac{2}{5} + \frac{1}{16} - \frac{1}{4} = \frac{57}{80},$$

$$(3.1) \geq \frac{\sqrt{3}}{\pi} \cdot \frac{57}{80} = 0,39 \ldots > \frac{1}{3}. \tag{3.17}$$

(4): $m_3 \geq 6$, $m_3 \in 3\mathbb{Z}$. Then $\widehat{F}(m_3) = -\frac{1}{2}$, $\widehat{F}(m_3-1) \geq -\frac{1}{m_3-1} + \frac{1}{2(m_3-1)} = -\frac{1}{2(m_3-1)}$, $\widehat{F}(m_3+1) \geq -\frac{1}{2} - \frac{1}{2(m_3+1)}$ and

$$(3.18) \qquad (3.14) \geq \frac{1}{2} + \frac{1}{2} - \frac{1}{4(m_3-1)} - \frac{1}{4} - \frac{1}{4(m_3+1)} \geq \frac{3}{4} - \frac{1}{20} - \frac{1}{28},$$

$$(3.19) \qquad (3.1) \geq \frac{\sqrt{3}}{\pi} \cdot \left(\frac{3}{4} - \frac{1}{20} - \frac{1}{28}\right) = 0,36\ldots > \frac{1}{3}.$$

(5): $m_3 \geq 6$, $m_3 \in 3\mathbb{Z}+1$. Then $\widehat{F}(m_3) \leq -\frac{1}{2} + \frac{1}{m_3} - \frac{1}{2m_3} = -\frac{1}{2} + \frac{1}{2m_3}$, $\widehat{F}(m_3-1) = 0$, $\widehat{F}(m_3+1) \geq -\frac{1}{2} - \frac{1}{m_3+1} + \frac{1}{2(m_3+1)} = -\frac{1}{2} - \frac{1}{2(m_3+1)}$ and

$$(3.14) \geq \frac{1}{2} + \frac{1}{2} - \frac{1}{2m_3} - \frac{1}{4} - \frac{1}{4(m_3+1)}$$

$$(3.20) \qquad = \frac{3}{4} - \frac{1}{2m_3} - \frac{1}{4(m_3+1)} \geq \frac{3}{4} - \frac{1}{14} - \frac{1}{32},$$

$$(3.21) \qquad (3.1) \geq \frac{\sqrt{3}}{\pi} \left(\frac{3}{4} - \frac{1}{14} - \frac{1}{32}\right) = 0,35\ldots > \frac{1}{3}.$$

(6): $m_3 \geq 6$, $m_3 \in 3\mathbb{Z}+2$. Then $\widehat{F}(m_3) = -\frac{1}{2} + \frac{1}{2m_3}$, $\widehat{F}(m_3-1) \geq -\frac{1}{2(m_3-1)}$, $\widehat{F}(m_3+1) \geq -\frac{1}{2}$ and

$$(3.14) \geq \frac{1}{2} + \frac{1}{2} - \frac{1}{2m_3} - \frac{1}{4(m_3-1)} - \frac{1}{4} =$$

$$(3.22) \qquad \frac{3}{4} - \frac{1}{2m_3} - \frac{1}{4(m_3-1)} \geq \frac{3}{4} - \frac{1}{16} - \frac{1}{28},$$

$$(3.23) \qquad (3.1) \geq \frac{\sqrt{3}}{\pi} \left(\frac{3}{4} - \frac{1}{16} - \frac{1}{28}\right) = 0,35\ldots > \frac{1}{3}.$$

From (3.9), (3.12), (3.15), (3.16), (3.17), (3.19), (3.21), (3.23), it follows that for any $B \subset \mathbb{Z}_+$, $|B| \geq 3$

$$(3.24) \qquad \max_{x \in \mathbb{T}} \left[\sum_{m \in B} \left(f - \frac{1}{3}\right)(mx)\right] > \frac{1}{3}$$

hence, by (2.1),

$$S(B) > \frac{|B|}{3} + \frac{1}{3},$$

$$S(B) \geq \frac{|B|}{3} + \frac{2}{3},$$

proving Proposition 1.3.

4. Poof of Proposition 1.4

Since

$$F(x) = \sum_{m\in B}\left(f - \frac{1}{3}\right)(mx)$$

satisfies $\int_{\mathbb{T}} F\, dx = 0$, it follows that

$$\max_{x\in\mathbb{T}} F(x) \geq \frac{1}{2}\|F\|_{L^1(\mathbb{T})}. \tag{4.1}$$

From equality (2.6)

$$\frac{\sqrt{3}}{\pi}\left\|\sum_{m\in B}\cos mx\right\|_1 \leq$$

$$\left[\prod_{p\leq P, p\neq 3}\left(1+\frac{1}{p}\right)\right]\|F\|_1 + \frac{\sqrt{3}}{\pi}\left\|\sum_{\substack{n\in\mathcal{G}_{2,\dots,P}, n>1\\ m\in B}}\frac{\chi(n)}{n}\cos mnx\right\|_1 . \tag{4.2}$$

The second term in (4.2) is bounded by

$$\frac{\sqrt{3}}{\sqrt{2}\pi}\cdot|B|\left[\sum_{\substack{n\in\mathcal{G}_{2,\dots,P}\\ m\in B}}\left(\frac{\chi(n)}{n}\right)^2\right]^{1/2} < C\,|B|\,P^{-1/2}\,. \tag{4.3}$$

Taking $P = |B|^2$, it follows that

$$\|F\|_1 > c\prod_{p\leq P}\left(1+\frac{1}{p}\right)^{-1}\cdot\left\|\sum_{m\in B}\cos mx\right\|_1 > c(\log P)^{-1}\left\|\sum_{m\in B}\cos mx\right\|_1 \tag{4.4}$$

and this proves Proposition 1.4.

5. Further estimates on (2.6)

LEMMA 5.1: *Let S be a finite subset of $\mathbb{Z}_+$ and $P > (\log|S|)^2$. Then for $K > 1$*

$$\frac{1}{K}\sum_{\substack{k\leq K\\ k\in\mathcal{G}_{2,3,\dots,P}}}|S\cap kS| < C\,P^{-1/2}\,|S|\cdot\log|S|. \tag{5.2}$$

Proof: (i) Construction of a partition of $I \equiv \{k\leq K \mid k\in\mathcal{G}_{2,3,\dots,P}\}$. For $k\in I$, denote $q(k)$ the largest prime divisor. Hence $q(k)\geq P$. Define

$$I_n = \{k\in I \mid k = n\,q(k)\}\ .$$

Hence $I = \bigcup_{n \le K/P} I_n$, $\sum_{n \le K/P} |I_n| < K$.

Assume

$$\frac{1}{K} \sum_{k \in I} |S \cap kS| > \delta |S|. \tag{5.3}$$

Then

$$\frac{1}{K} \sum_{n \le \frac{K}{P}} |I_n| \left(\frac{1}{|I_n|} \sum_{k \in I_n} |S \cap kS| \right) > \delta |S|.$$

Hence, there is $n' \le K/P$ such that $I' = I_{n'}$ satisfies

$$|I'| > \frac{\delta}{10} P \quad \text{and} \quad \frac{1}{|I'|} \sum_{k \in I'} |S \cap kS| > \frac{\delta}{10} |S|. \tag{5.4}$$

Denote $J = \frac{1}{n'} I'$ the corresponding set of primes. Thus

$$|J| > \frac{\delta}{10} P \quad \text{and} \quad \frac{1}{|J|} \sum_{p \in J} |S \cap n'pS| > \frac{\delta}{10} |S|. \tag{5.5}$$

(ii) Define next

$$\begin{aligned} S_\tau^+ &= \{x \in S \mid \#\{p \in J \mid n'px \in S\} > \tau |J|\}, \\ S_\tau^- &= \left\{x \in S \mid \#\{p \in J \mid n'p \mid x \quad \text{and} \quad \frac{x}{n'p} \in S\} > \tau |J| \right\}, \\ S_\tau &= S_\tau^+ \cup S_\tau^-. \end{aligned}$$

Hence

$$\frac{1}{|J|} \sum_{p \in J} |\, (S \backslash S_\tau) \cap n'pS| =$$

$$\frac{1}{|J|} \sum_{x \in S \backslash S_\tau} \# \left\{ p \in J \mid n'p \mid x \quad \text{and} \quad \frac{x}{n'p} \in S \right\} < \tau |S| \tag{5.6}$$

and similarly

$$\frac{1}{|J|} \sum_{p \in J} |S \cap n'p(S \backslash S_\tau)| = \frac{1}{|J|} \sum_{x \in S \backslash S_\tau} \# \{p \in J \mid n'px \in S\} < \tau |S|. \tag{5.7}$$

Thus (5.5), (5.6), (5.7) imply

$$\frac{1}{|J|} \sum_{p \in J} |S_\tau \cap n'pS_\tau| > \left(\frac{\delta}{10} - 2\tau \right) |S_\tau|. \tag{5.8}$$

Fix an integer $r \sim \log|S|$ and define $\tau = \delta/100r$. With previous notation, define the decreasing sequence of sets

$$S^0 = S,\ S^1 = S_\tau,\ S^{q+1} = (S^q)_\tau \quad \text{for } q < r.$$

By definition of τ

$$\frac{1}{|J|}\sum_{p\in J}|S^q \cap n'pS^q| > \frac{\delta}{20}|S^q| \quad \text{for } q \le r \tag{5.9}$$

and in particular

$$S^r \neq \emptyset. \tag{5.10}$$

(iii) Construction of a tree. Take a point $x \in S^r$. We will introduce points $x(\ell_1, \ell_2, \dots, \ell_w)$ with $\ell_i = 1, 2, \dots, 10$ and $w < r$.

Since $x \in S^r = (S^{r-1})_\tau$ there is $\varepsilon_0 = \pm 1$ such that, by (5.5) (and (5.18) below),

$$\#\left\{p \in J \mid (n'p)^{\varepsilon_0} x \in S^{r-1}\right\} > \tau \cdot |J| = \frac{\delta^2}{10^3 r}\, P > 10. \tag{5.11}$$

Choose $p(\ell_1)$, $\ell_1 = 1, \dots, 10$ with $(n'p(\ell_1))^{\varepsilon_0} x \in S^{r-1}$ and define

$$x(\ell_1) = (n'p(\ell_1))^{\varepsilon_0}\, x. \tag{5.12}$$

If the construction is performed up to stage w with $x(\ell_1, \dots, \ell_w) \in S^{r-w}$, we proceed as follows. Since for some $\varepsilon = \varepsilon(\ell_1, \dots, \ell_w) = \pm 1$,

$$\#\left\{p \in J \mid (n'p)^\varepsilon\, x(\ell_1, \dots, \ell_w) \in S^{r-w-1}\right\} > \frac{\delta^2 P}{10^3 r} > 10w, \tag{5.13}$$

we may find distinct $p(\ell_1, \dots, \ell_w, \ell_{w+1})$, $\ell_{w+1} = 1, \dots, 10$ satisfying

$$(n'p(\ell_1, \dots, \ell_{w+1}))^\varepsilon\, x(\ell_1, \dots, \ell_w) \in S^{r-w-1} \tag{5.14}$$

and

$$\begin{aligned} &p(\ell_1, \dots, \ell_{w+1}) \notin \{p(\ell) \mid \ell = 1, \dots, 10\} \cup \\ &\{p(\ell_1, \ell) \mid \ell = 1, \dots, 10\} \cup \dots \cup \{p(\ell_1, \dots, \ell_{w-1}, \ell) \mid \ell = 1, \dots, 10\}. \end{aligned} \tag{5.15}$$

Define

$$x(\ell_1, \dots, \ell_w, \ell_{w+1}) = [n'p(\ell_1, \dots, \ell_{w+1})]^{\varepsilon(\ell_1, \dots, \ell_w)}\ x(\ell_1, \dots, \ell_w). \tag{5.16}$$

Thus from (5.12), (5.16)

$$x(\ell_1,\dots,\ell_r) = [n'p(\ell_1)]^{\varepsilon_0}\,[n'p(\ell_1,\ell_2)]^{\varepsilon(\ell_1)}\dots[n'p(\ell_1,\dots,\ell_r)]^{\varepsilon(\ell_1,\dots,\ell_{r-1})}\ x. \tag{5.17}$$

To fulfil condition (5.13), we need

$$P > 10^4\ \delta^{-2}\ r^2\ . \tag{5.18}$$

(iv) For each $\overline{\varepsilon} = (\varepsilon_1,\dots,\varepsilon_{r-1}) \in \{1,-1\}^{r-1}$, let

$$\alpha_{\overline{\varepsilon}} = \Big\{(\ell_1,\ell_2,\dots,\ell_r) \in \{1,\dots,10\}^r \mid \varepsilon(\ell_1) = \varepsilon_1\ ,\ \varepsilon(\ell_1,\ell_2) = \varepsilon_{2,\dots}\ ,\ \varepsilon(\ell_1,\dots,\ell_{r-1}) = \varepsilon_{r-1}\Big\}. \tag{5.19}$$

Thus we may fix some $\overline{\varepsilon}$ such that

$$\#\ \alpha_{\overline{\varepsilon}} > 10^r\ 2^{-(r-1)} > 2^r. \tag{5.20}$$

We claim that the points

$$x(\ell_1,\dots,\ell_r) \quad \text{with } (\ell_1,\dots,\ell_r) \in \alpha_{\overline{\varepsilon}} \tag{5.21}$$

are all distinct and thus by (5.20)

$$|S| > 2^r. \tag{5.22}$$

By construction and (5.17)

$$x(\ell_1,\dots,\ell_r) = (n')^{\varepsilon_0+\varepsilon_1+\dots+\varepsilon_{r-1}}\ p(\ell_1)^{\varepsilon_0}\ p(\ell_1,\ell_2)^{\varepsilon_1}\dots p(\ell_1,\dots,\ell_r)^{\varepsilon_{r-1}}\ x \tag{5.23}$$

and the claim will result from the fact that for given $\overline{\varepsilon}$ all rationals

$$p(\ell_1)^{\varepsilon_0}\ p(\ell_1,\ell_2)^{\varepsilon_1}\dots p(\ell_1,\dots,\ell_r)^{\varepsilon_{r-1}}, \quad \ell_i = 1,\dots,10$$

are distinct. Now this property is easily seen to result from (5.15).

In order to contradict (5.22), we take thus $r \sim \log|S|$. Condition (5.18) yields then $\delta > 10^2 \cdot \log|S| \cdot P^{-1/2}$, proving the lemma.

Remark: Observe that (for K large wrt P) one has

$$\# \{k \le K \mid k \in \mathcal{G}_{2,\dots,P}\} \sim \prod_{p \le P} \left(1 - \frac{1}{p}\right) K \sim (\log P)^{-1} K \tag{5.24}$$

which does not imply (5.2).

LEMMA 5.25: *Let* $|S| = N$, $P > (\log N)^4$. *Let* $f \in L^2(\mathbb{T})$, $\|f\|_2 \le 1$. *Then*

$$\frac{1}{K} \sum_{\substack{k \le K \\ 1 < k \in \mathcal{G}_{2,\dots,P}}} \left\| \widehat{f} \,|_{kS} \right\|_{\ell^1(\mathbb{Z})} < P^{-1/8} N^{1/2}. \tag{5.26}$$

Proof: Consider level sets $A_\gamma = \left\{n \in \mathbb{Z} \mid |\widehat{f}(n)| \sim \gamma\right\}$. Thus

$$|A_\gamma| \le \gamma^{-2}. \tag{5.27}$$

The left member of (5.26) equals

$$\frac{1}{K} \sum_{\substack{k \le K \\ k \in \mathcal{G}_{2,\dots,P}}} \sum_{\substack{0 < \gamma < 1 \\ \gamma \text{ dyadic}}} \gamma |A_\gamma \cap kS|. \tag{5.28}$$

We distinguish 3 cases.

(I) $\gamma < N^{-2}$. Estimate the corresponding summand in (5.28) trivially by

$$N \cdot N^{-2} < N^{-1}.$$

(II) $N^{-1/2} > \gamma > N^{-2}$. Fix γ and let $S^\gamma = S \cup A_\gamma$. Hence $|S^\gamma| \le \gamma^{-2}$. Thus, by Lemma 5.1

$$\frac{1}{K} \sum_{\substack{k \le K \\ k \in \mathcal{G}_{2,\dots,P}}} |A_\gamma \cap kS| \le \frac{1}{K} \sum_{\substack{k \le K \\ k \in \mathcal{G}_{2,\dots,P}}} |S^\gamma \cap kS^\gamma| < \frac{\gamma^{-2}}{P^{1/4}} \tag{5.29}$$

and hence

$$\frac{\gamma}{K} \sum_{\substack{k \le K \\ k \in \mathcal{G}_{2,\dots,P}}} |A_\gamma \cap kS| < \min\left(\frac{\gamma^{-1}}{P^{1/4}}, \gamma N\right) \tag{5.30}$$

which contribution is

$$\le \frac{N^{1/2}}{P^{1/8}}. \tag{5.31}$$

(III) $\gamma > N^{-1/2}$. In this case $|S^\gamma| \leq N$ and by Lemma 5.1

$$\frac{1}{K} \sum_{\substack{k \leq K \\ k \in \mathcal{G}_{2,\dots,P}}} |A_\gamma \cap kS| \leq \frac{N}{P^{1/4}} \tag{5.32}$$

and

$$\frac{\gamma}{K} \sum_{\substack{k \leq K \\ k \in \mathcal{G}_{2,\dots,P}}} |A_\gamma \cap kS| < \min\left(\gamma^{-1}, \gamma \frac{N}{P^{1/4}}\right) \tag{5.33}$$

which contribution is

$$\leq \frac{N^{1/2}}{P^{1/8}}. \tag{5.34}$$

This proves Lemma 5.25.

Lemma 5.35: *Let B be a finite subset of $\mathbb{Z}_+$ and define for all $R \geq 1$*

$$B_R = \{m \in B \mid m < R\}. \tag{5.36}$$

Let $|a_n| \leq 1$ and $P > (\log |B|)^{20}$. Then

$$\left\| \mathrm{Proj}_{[0,R]} \left[\sum_{1 < n \in \mathcal{G}_{2,\dots,P}, m \in B} \frac{a_n}{n} e^{imnx} \right] \right\|_2 < P^{-1/15} |B_R|^{1/2}. \tag{5.37}$$

Proof: Observe first that if $m \in B$, $n > 1$ and $mn < R$, then necessarily $m \in B_R$. The left member of (5.37) may thus be estimated by

$$\sum_{\substack{K > 1 \\ K \text{dyadic}}} K^{-1} \left\| \sum_{\substack{1 < n \in \mathcal{G}_{2,\dots,P}, n \sim K \\ m \in B_R}} a_n e^{imnx} \right\|_2 \tag{5.38}$$

making a dyadic partition in n. From Lemma 5.25 and duality, it follows that

$$K^{-1} \left\| \sum_{\substack{1 < n \in \mathcal{G}_{2,\dots,P}, n \sim K \\ m \in B_R}} a_n e^{imnx} \right\|_2 < C\, P^{-1/8}\, |B_R|^{1/2}. \tag{5.39}$$

Also, there is the trivial estimate $|B_R| K^{-1/2}$ on the left side of (5.39). Inequality (5.37) follows.

6. The Littlewood conjecture revisited

In this section we recall the proof of the Littlewood conjecture

$$\left\| \sum_{n=1}^{N} e^{im_n x} \right\|_{L^1(\mathbb{T})} > c \log N \tag{6.1}$$

for

$$m_1 < m_2 < \cdots < m_N \in \mathbb{Z}$$

due to [M-P-S], with a few adjustments needed later on.

Let $B = \{m_1 < m_2 < \cdots < m_N\} \subset \mathbb{Z}_+$. Denote

$$F = \sum_{n=1}^{N} e^{im_n x} \tag{6.2}$$

and $r_0 \sim \log N$. The inequality (6.1) is obtained using a test polynomial Φ of the form

$$\Phi = P_1 + Q_1 P_2 + Q_1 Q_2 P_3 + \cdots + Q_1 \cdots Q_{r_0 - 1}\, P_{r_0} \tag{6.3}$$

where $|\Phi| < C$ and for each r

$$\langle F, Q_1 \cdots Q_{r-1}\, P_r \rangle > c. \tag{6.4}$$

Hence

$$\langle F, \Phi \rangle = \sum_{r \le r_0} \langle F, Q_1 \cdots Q_{r-1} P_r \rangle \sim r_0. \tag{6.5}$$

We will need some information on the Fourier transform of the Q_r-functions, which will require a slight modification of the original construction. More precisely, we will require

$$\operatorname{supp} \widehat{P}_r \subset \left\{ m_j \mid N 10^{-6r} < j < N 10^{-6(r-1)} \right\} \tag{6.6}$$

and

$$\operatorname{supp} \widehat{Q}_r \subset \left[-\frac{1}{10} (m_{N \cdot 10^{-6(r-1)}} - m_{N \cdot 10^{-6r}}), 0 \right]. \tag{6.7}$$

This may be achieved as follows. Partition the interval

$$I =]m_{N 10^{-6r}}, m_{N 10^{-6(r-1)}}[$$

in 40 intervals I_α of size $\frac{1}{40}|I|$ and select α such that

$$N10^{-6(r-1)} > X_r \equiv \#\{j \mid m_j \in I_\alpha\} > \frac{1}{50}\, N10^{-6(r-1)}. \tag{6.8}$$

Define

$$P_{r,1} = \frac{1}{X_r} \sum_{m_j \in I_\alpha} e^{im_j x}, \tag{6.9}$$

$$P_r = P_{r,1} * \left(e^{i\xi_\alpha x} F_{2|I_\alpha|}\right), \tag{6.10}$$

where ξ_α is the center of I_α and $F_M = \sum_{|m|\leq M} \frac{M-|m|}{M}\, e^{imx}$ is the M-Féjer kernel.

Hence

$$\operatorname{supp} \widehat{P}_r \subset \xi_\alpha + [-2\,|I_\alpha|, 2\,|I_\alpha|] \tag{6.11}$$

and

$$\langle F, P_r\rangle > \frac{1}{2}. \tag{6.12}$$

Define

$$Q_r = \left[e^{-(|P_{r,1}|+i\mathcal{H}[|P_{r,1}|])}\right] * F_{2|I_\alpha|} \tag{6.13}$$

where $\mathcal{H}$ denotes the Hilbert transform on $L^2(\mathbb{T})$.

Hence

$$\operatorname{supp} \widehat{Q}_r \subset [-2|I_\alpha|, 0]$$

implying (6.7). Observe that by construction

$$\begin{aligned} \left\|\frac{1}{10}\,|P_r| + |Q_r|\right\|_\infty &\leq \left\|\left[\frac{1}{10}\,|P_{r,1}| + e^{-|P_{r,1}|}\right] * F_{2|I_\alpha|}\right\|_\infty \\ &\leq \left\|\frac{1}{10}\,|P_{r,1}| + e^{-|P_{r,1}|}\right\|_\infty \\ &\leq 1. \end{aligned} \tag{6.14}$$

Hence, $|\Phi| < 10$ by iteration of (14), since $|P_r| \leq 1$.

By (6.8)

$$\begin{aligned} \|1 - Q_r\|_2 &\leq \|\, |P_{r,1}| + |\mathcal{H}(|P_{r,1}|)|\, \|_2 \leq 2\|P_{r,1}\|_2 \\ &= 2X_r^{-1/2} < 16N^{-1/2}\, 10^{+3(r-1)}. \end{aligned} \tag{6.15}$$

Write thus, using (6.7), (6.12),

$$\begin{aligned}\langle F,\Phi\rangle &= \sum_{r\le r_0}\langle F,P_r\rangle + \sum_{r\le r_0}\langle F,P_r(1-Q_1\cdots Q_{r-1})\rangle\\ &\ge \frac{r_0}{2} + \sum_{r\le r_0}\langle F_r,P_r(1-Q_1\cdots Q_{r-1})\rangle \end{aligned}\tag{6.16}$$

denoting

$$F_r = \sum_{j\le N10^{-6(r-1)}} e^{im_jx}\ . \tag{6.17}$$

From (6.15)

$$\begin{aligned}|\langle F_r,P_r(1-Q_1\cdots Q_{r-1})\rangle| &\le \|F_r\|_2\sum_{s=1}^{r-1}\|1-Q_s\|_2\\ &\le N^{1/2}10^{-3(r-1)}\sum_{s=1}^{r-1}16N^{-1/2}10^{3(s-1)} < \frac{1}{10}\end{aligned}\tag{6.18}$$

and (6.16) yields thus

$$\|F\|_1 \ge \frac{|\langle F,\Phi\rangle|}{10} > c\log N\ . \tag{6.19}$$

Next, letting $P > (\log N)^{100}$, $|a_n|\le 1$, we need estimates on

$$\Big\langle \sum_{\substack{1<n\in\mathcal{G}_{2,\dots,P}\\ m\in B}} \frac{a_n}{n}e^{imnx},\Phi\Big\rangle, \tag{6.20}$$

$$\Big\langle \sum_{\substack{1<n\in\mathcal{G}_{2,\dots,P}\\ m\in B}} \frac{a_n}{n}e^{-imnx},\Phi\Big\rangle. \tag{6.21}$$

LEMMA 6.22: *With previous notations*

$$|(6.20)| + |(6.21)| < C\ P^{-1/15}\ (\log|B|)^2.$$

Write

$$(6.20) \le \sum_{r\le r_0}|\langle G,P_r\rangle| + \sum_{\substack{r\le r_0\\ 1\le s<r}}|\langle G,P_r(1-Q_s)Q_{s+1}\cdots Q_{r-1}\rangle| \tag{6.22}$$

with $G = \sum_{\substack{1<n\in\mathcal{P}_{2,\dots,P}\\ m\in B}} \frac{a_n}{n}\, e^{imnx}$.

Observe that by (6.6), (6.7)

$$\operatorname{supp}[P_r(1-Q_s)\,Q_{s+1}\dots Q_r]^\wedge \subset \Big[m_{N10^{-6(r-1)}}, -\frac{1}{10}\big(m_{N\cdot 10^{-6(s-1)}} - m_{N\cdot 10^{-6s}} + m_{N\cdot 10^{-6s}} - m_{N\cdot 10^{-6(s+1)}} \dots - m_{N\cdot 10^{-6r}}\big)\Big]$$

$$\subset \big[m_{N\cdot 10^{-6(r-1)}},\ -\frac{1}{10}\ m_{N\cdot 10^{-6(s-1)}}\ \big]. \tag{6.23}$$

By (5.37), taking $R = m_{N\cdot 10^{-6(r-1)}}$, it follows from (6.15) that

$$\begin{aligned}|\langle G, P_r(1-Q_s)Q_{s+1}\dots Q_{r-1}\rangle| \leq& P^{-1/15}N^{1/2}\cdot 10^{-3(r-1)} \\ & \|P_r(1-Q_s)Q_{s+1}\dots Q_{r-1}\|_2 \\ \leq& P^{-1/15}N^{1/2}10^{-3(r-1)}\cdot 16N^{-1/2}10^{3(s-1)} \\ \leq& P^{-1/15}10^{3(s-r)}.\end{aligned} \tag{6.24}$$

Hence

$$(6.20) \leq CP^{-1/15}r_0. \tag{6.25}$$

Write similarly

$$\begin{aligned}(6.21) \leq& \sum_{\substack{r\leq r_0\\ 1\leq s<r}} |\langle \overline{G}, P_r(1-Q_s)Q_{s+1}\cdots Q_{r-1}\rangle| \\ \leq& \sum_{\substack{r\leq r_0\\ s<r}} CP^{-1/15}\left(N\cdot 10^{-6(s-1)}\right)^{1/2} \|P_r(1-Q_s)Q_{s+1}\cdots Q_{r-1}\|_2 \\ <& CP^{-1/15}r_0^2\end{aligned} \tag{6.26}$$

applying now (5.37) with $R = m_{N\cdot 10^{-6(s-1)}}$.

This proves Lemma 6.22.

As a consequence of the preceding, one gets by (6.5), (6.22),

Lemma 6.27: *Let B be a finite subset of $\mathbb{Z}_+$ and $P > (\log|B|)^{100}$. Let $|a_n|, |b_n| \leq 1$. Then*

$$\left\|\sum_{m\in B} e^{imx} + \sum_{\substack{1<n\in\mathcal{P}_{2,\dots,P}\\ m\in B}} \frac{1}{n}\ \left(a_n\ e^{imnx} + b_n\ e^{-imnx}\right)\right\|_1 > c\log|B|.$$

7. Proof of Proposition 1.7

We use a similar approach as for the minoration of $s_2(B)$.

Consider

$$f_+ = \mathbb{I}_{]\frac{1}{4}-\frac{1}{8},\frac{1}{4}+\frac{1}{8}[} \quad \text{and} \quad f_- = \mathbb{I}_{]-\frac{1}{4}-\frac{1}{8},-\frac{1}{4}+\frac{1}{8}[}. \tag{7.1}$$

Then

$$S_3(B) \geq \frac{|B|}{4} + \max_{x\in\mathbb{T}} \left[\sum_{m\in B}\left(f_\pm - \frac{1}{4}\right)(mx)\right]. \tag{7.2}$$

Fourier expansion yields

$$\left(f_\pm - \frac{1}{4}\right)(x) = \sum_{n\neq 0} \frac{1}{n\pi} \sin \frac{n\pi}{4}\, e^{\pm i\frac{\pi}{2}n}\, e^{2\pi inx} \tag{7.3}$$

and hence

$$\begin{aligned}\left(f_+ + f_- - \frac{1}{2}\right)(x) &= \frac{4}{\pi}\sum_{n\geq 1}\frac{1}{n}\sin\frac{n\pi}{4}\cdot\cos n\frac{\pi}{2}\cdot\cos 2\pi nx\\ &= -\frac{2}{\pi}\sum\frac{1}{n}\cdot\sin n\frac{\pi}{2}\cdot\cos 4\pi nx\\ &= -\frac{2}{\pi}\sum\frac{1}{n}\chi_1(n)\cos 4\pi nx \end{aligned} \tag{7.4}$$

where χ_1 is multiplicative mod (4).

Similarly

$$\begin{aligned}(f_+ - f_-)(x) &= -\frac{4}{\pi}\sum_{n\geq 1}\frac{1}{n}\sin\frac{n\pi}{4}\sin\frac{n\pi}{2}\sin 2\pi nx\\ &= -\frac{2\sqrt{2}}{\pi}\sum_{n\geq 1}\frac{1}{n}\chi_2(n)\sin 2\pi nx\end{aligned} \tag{7.5}$$

where χ_2 is the multiplicative character mod (8) defined by

$$\begin{cases}\chi_2(0) = 0\\ \chi_2(1) = 1\\ \chi_2(2) = 0\\ \chi_2(3) = -1\\ \chi_2(4) = 0\\ \chi_2(5) = -1\\ \chi_2(6) = 0\\ \chi_2(7) = 1\end{cases}. \tag{7.6}$$

In analogy to the identity (2.6), we get thus

$$(7.7)\qquad \begin{aligned}&\sum_{k|P!}\frac{\mu(k)}{k}\chi_1(k)\left[\sum_{m\in B}\left(f_+ + f_- - \frac{1}{2}\right)(mkx)\right] =\\ &-\frac{2}{\pi}\sum_{m\in B}\cos 2mx - \frac{2}{\pi}\sum_{\substack{1<n\in\mathcal{G}_{2,\dots,P}\\ m\in B}}\frac{\chi_1(n)}{n}\cos 2mnx\end{aligned}$$

and

$$(7.8)\qquad \begin{aligned}&\sum_{k|P!}\frac{\mu(k)}{k}\chi_2(k)\left[\sum_{m\in B}(f_+ - f_-)(mkx)\right] =\\ &-\frac{2\sqrt{2}}{\pi}\sum_{m\in B}\sin mx - \frac{2\sqrt{2}}{\pi}\sum_{\substack{1<n\in\mathcal{G}_{2,\dots,P}\\ m\in B}}\frac{\chi_2(n)}{n}\sin mnx.\end{aligned}$$

Taking $P\sim(\log|B|)^{100}$ and applying (6.27) yields from (7.7), (7.8)

$$\begin{aligned}c\log|B| &< \left\|\sum_{m\in B}e^{2imx} + \sum_{\substack{1<n\in\mathcal{G}_{2,\dots P}\\ m\in B}}\frac{1}{n}\left(\chi_1(n)\cos 2mnx + i\,\chi_2(n)\sin 2mnx\right)\right\|_1\\ &\le\sum_{k|P!}\frac{|\mu(k)|}{k}\left[\frac{\pi}{2}\left\|\sum_{m\in B}\left(f_+ + f_- - \frac{1}{2}\right)(mkx)\right\|_1\right.\\ &\qquad\left. + \frac{\pi}{2\sqrt{2}}\left\|\sum_{m\in B}(f_+ - f_-)(mkx)\right\|_1\right]\end{aligned}$$

$$(7.9)\qquad \le c\prod_{\substack{p\le P\\ p\,\mathrm{prime}}}\left(1+\frac{1}{p}\right)\left[\left\|\sum_{m\in B}\left(f_+ - \frac{1}{4}\right)(mx)\right\|_1 + \left\|\sum_{m\in B}\left(f_- - \frac{1}{4}\right)(mx)\right\|_1\right].$$

Consequently,

$$(7.10)\qquad \max_{x\in\mathbb{T}}\left[\sum_{m\in B}\left(f_\pm - \frac{1}{4}\right)(mx)\right] > c\frac{\log|B|}{\log P} = c\frac{\log|B|}{\log\log|B|}$$

which by (7.2) implies Proposition 1.7.

Remarks:

1. The problem to carry out the preceding to improve the lower estimate on $s_2(B)$ is the fact that if we consider (instead of $\mathbb{I}_{[\frac{1}{2}-\frac{1}{6},\frac{1}{2}+\frac{1}{6}]}$) a set $\Omega\subset\mathbb{T}$ satisfying $|\Omega| = \frac{1}{3}$, $(\Omega+\Omega)\cap\Omega = \emptyset$, then necessarily (by Kneser's theorem), $\Omega = \mathbb{T}\backslash(\Omega-\Omega)$ and hence is symmetric.
2. The proof of Proposition 1.7 has similarities with [B].

8. Further remarks

Call $A \subset \mathbf{Z}_+$ k-sum free provided

$$\underbrace{A + \cdots + A}_{k \text{ times}} \equiv (A)_k \cap A = \emptyset. \tag{8.1}$$

Denote $s_k(A)$ the cardinality of a maximum size subset B of A which is k-sum free. One gets easily

$$s_k(A) > \frac{1}{k+1}|A|. \tag{8.2}$$

In fact, there is also the following infinite version of this property:

Let $\{n_j\}$ be an increasing sequence in $\mathbb{Z}_+$. There is a set S such that $\{n_j | j \in S\}$ is k sum-free and

$$\lim_N \frac{1}{N}|S \cap [0, N]| \geq \frac{1}{k+1}. \tag{8.3}$$

Both facts (8.2), (8.3) are easily obtained considering the analogue of (2.1), with now $J = \left[\frac{1}{2(k-1)} - \frac{1}{2(k+1)}, \frac{1}{2(k-1)} + \frac{1}{2(k+1)}\right]$, averaging over $x \in \mathbb{T}$.

Our next purpose is to show that an estimate

$$s_k(A) > \delta_k |A| \tag{8.4}$$

for any finite $A \subset \mathbb{Z}_+$ requires at least $\delta_k \to 0$ for $k \to \infty$. This fact will follow by constructing appropriate examples.

We will rely on the following fact, which is an exercise on the circle method (we omit the proof).

LEMMA 8.5: *Let $A \subset \mathbb{Z} \cap [0, M]$ and assume*

$$|A| > \delta M \tag{8.6}$$

(with $\delta > 0$ a fixed constant). Thus there are integers r and q and an interval I such that

$$r, q < C(\delta), \tag{8.7}$$

$$|I| = M, \quad I \subset [M, rM], \tag{8.8}$$

$$q(I \cap \mathbb{Z}) \subset (A)_r. \tag{8.9}$$

We make the following construction.

Fix N, J (to be specified depending on k, δ). Denote

$$A_j = \{j!n \mid \frac{N}{2} \le n \le N\}, \tag{8.10}$$

$$A = \bigcup_{j=1}^{J} A_j. \tag{8.11}$$

Assume $B \subset A$ and $|B| > \delta|A|$. We assume $\delta > 0$ fixed and will denote by C various constants that may depend on δ.

One may find integers $1 \le j_0 < j_1 < j_2 < J$ such that

$$j_1 - j_0 > \frac{1}{2}\delta J, \tag{8.12}$$

$$|j_1 - j_2| < J_1, \tag{8.13}$$

$$|B \cap A_{j_0}| > \frac{\delta}{4}N, \tag{8.14}$$

$$|B \cap A_{j_1}| > \frac{\delta}{4}N, \tag{8.15}$$

$$B \cap A_{j_2} \neq \emptyset. \tag{8.16}$$

Here $J_1 \ll J$, $\delta J_1 > C$, are to be specified. This follows easily from the fact that $|B| > \delta|A|$. From Lemma 8.5, (8.14), (8.15), there are integers r_0, q_0, r_1, q_1 and intervals I_0, I_1 of integers satisfying

$$r_0, r_1, q_0, q_1 < C, \tag{8.17}$$

$$I_0 \subset [N, r_0 N], \quad |I_0| = N, \tag{8.18}$$

$$I_1 \subset [N, r_1 N], \quad |I_1| = N, \tag{8.19}$$

$$\{q_1 j_0! n \mid n \in I_0\} \subset (B \cap A_{j_0})_{r_0}, \tag{8.20}$$

$$\{q_1 j_1! n \mid n \in I_1\} \subset (B \cap A_{j_1})_{r_1}. \tag{8.21}$$

Let $I_1 = [a_1, b_1]$, thus $b_1 > (1 + \frac{1}{r_1})a_1, a_1 > N$. Fix also some element $\bar{n} j_1! \in B \cap A_{j_1}$.

Consider $C < \ell_1 < kC^{-1}$ and choose ℓ_0 such that

$$r_0\ell_0 + r_1\ell_1 < k, \tag{8.22}$$

$$|k - r_0\ell_0 - r_1\ell_1| < C. \tag{8.23}$$

Consider the following subset of $(B)_k$:

$$(B \cap A_{j_0})_{r_0\ell_0} + (B \cap A_{j_1})_{r_1\ell_1} + (k - r_0\ell_0 - r_1\ell_1)\overline{n}j_1! \tag{8.24}$$

which by (8.20), (8.21) contains the set

$$\{q_0j_0!n_0 + q_1j_1!n_1 + (k - r_0\ell_0 - r_1\ell_1)\overline{n}j_1! | n_0 \in (I_0)_{\ell_0}, n_1 \in (I_1)_{\ell_1}\}. \tag{8.25}$$

From (8.12), $q_0j_0!|j_1!$. Assume $k < e^{\delta J}$, hence

$$k \ll \frac{j_1!}{j_0!}. \tag{8.26}$$

One easily checks that (8.25) contains a set of the form

$$j_1!(\mathbb{Z} \cap [q_1\ell_1a_1 + CN, q_1\ell_1b_1 - CN]). \tag{8.27}$$

Letting ℓ_1 range as above, it follows that $(B)_k$ contains a set

$$j_1!(\mathbb{Z} \cap [Ca_1, \frac{k}{C}b_1]) \tag{8.28}$$

$$\supset j_1!(\mathbb{Z} \cap [CN, \frac{k}{C}N]). \tag{8.29}$$

To obtain that $(B)_k \cap B \neq \emptyset$, it suffices thus by (8.16) to verify that

$$A_{j_2} = \{nj_2! | \frac{N}{2} \leq n \leq N\} \subset j_1!(\mathbb{Z} \cap [CN, \frac{k}{C}N]). \tag{8.30}$$

This will be the case provided

$$\frac{N}{2}j_2! > CNj_1!, \tag{8.31}$$

$$Nj_2! < \frac{k}{C}Nj_1!. \tag{8.32}$$

(8.31) follows from $j_2 > j_1 > \frac{1}{2}\delta J$ and, by (8.13), (8.32), will be satisfied if

$$\frac{j_2!}{j_1!} < J^{j_2 - j_1} < J^{J_1} < k. \tag{8.33}$$

Thus sumarizing the resulting conditions on the different parameters, we find $\delta J_1 > C, k < e^{\delta J}, J^{J_1} < k$. For k sufficently large (depending on δ), these are clearly compatible.

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